Laguerre Polynomials

Audrey Holloman

Overview

- Named after Edmond Laguerre (French Mathematician)
- These polynomials are the solution of Laguerre's equation
  - $x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0,\qquad$ n being a positive integer
- Used to calculate numerically in Gaussian quadrature
  - $\int_0^\infty f(x)e^{-x}dx$
- Have a wide range of applications in quantum mechanics
  - Radial part of the solution of the Schrödinger equation for a one-electron atom
  - Used to describe the static Wigner functions of oscillator systems in phase space

Solution

- Use $y = \sum_{m=o}^{\infty} c_m x^{k+m}$, $\quad c_0 \neq 0$ (Frobenius method) to solve Laguerre's equation
- Differentiate the summation above to get $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ and substitute them into Laguerre's equation
  - $\sum_{m=0}^{\infty} c_m (k+m)^2 x^{k+m-1} - \sum_{m=0}^{\infty} c_m (k+m-n) x^{k+m} = 0$
- Equate the coefficient of x^{k+m-1} to zero
  - $c_m = \frac{(k+m-1-n)}{(k+m)^2} c_{m-1}$
- Consider the solution $(y)_{k=0}$
  - $y = \sum_{m=0}^{\infty} c_m x^m$, where $c_m = \frac{(m-1-n)}{m^2} c_{m-1} \quad (k=0)$

Solution Continued

- Substitute $m = 1,2,3,\dots,r$ to get c_r
  - $c_r = (-1)^r \frac{n(n-1)\dots(n-r+1)}{(r!)^2} c_0$ for $r \leq n$ and $c_{n+1} = c_{n+2} = c_{n+3} = \cdots = 0$
- Substitute this into the last summation
  - $y = c_0 \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!(r!)^2} x^r$
- Taking $c_0 = 1$, the solution is defined as the Laguerre polynomial of order n
  - $L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!(r!)^2} x^r$
  - The abscissas for quadrature order n are given by the roots of the Laguerre polynomials $L_n(x)$

Polynomials

- Use the solution to Laguerre's equation (i.e. $L_n(x)$) to find a few Laguerre polynomials
  - $L_0(x) = (-1)^0 \frac{0!}{(0-0)!(0!)^2} x^0 = 1$
  - $L_1(x) = (-1)^0 \frac{1!}{(1-0)!(0!)^2} x^0 + (-1)^1 \frac{1!}{(1-1)!(1!)^2} x^1 = 1 - x$

n	$L_n(x)$
0	1
1	$1 - x$
2	$\frac{1}{2}(x^2 - 4x + 2)$
3	$\frac{1}{6}(-x^3 + 9x^2 - 18x + 6)$

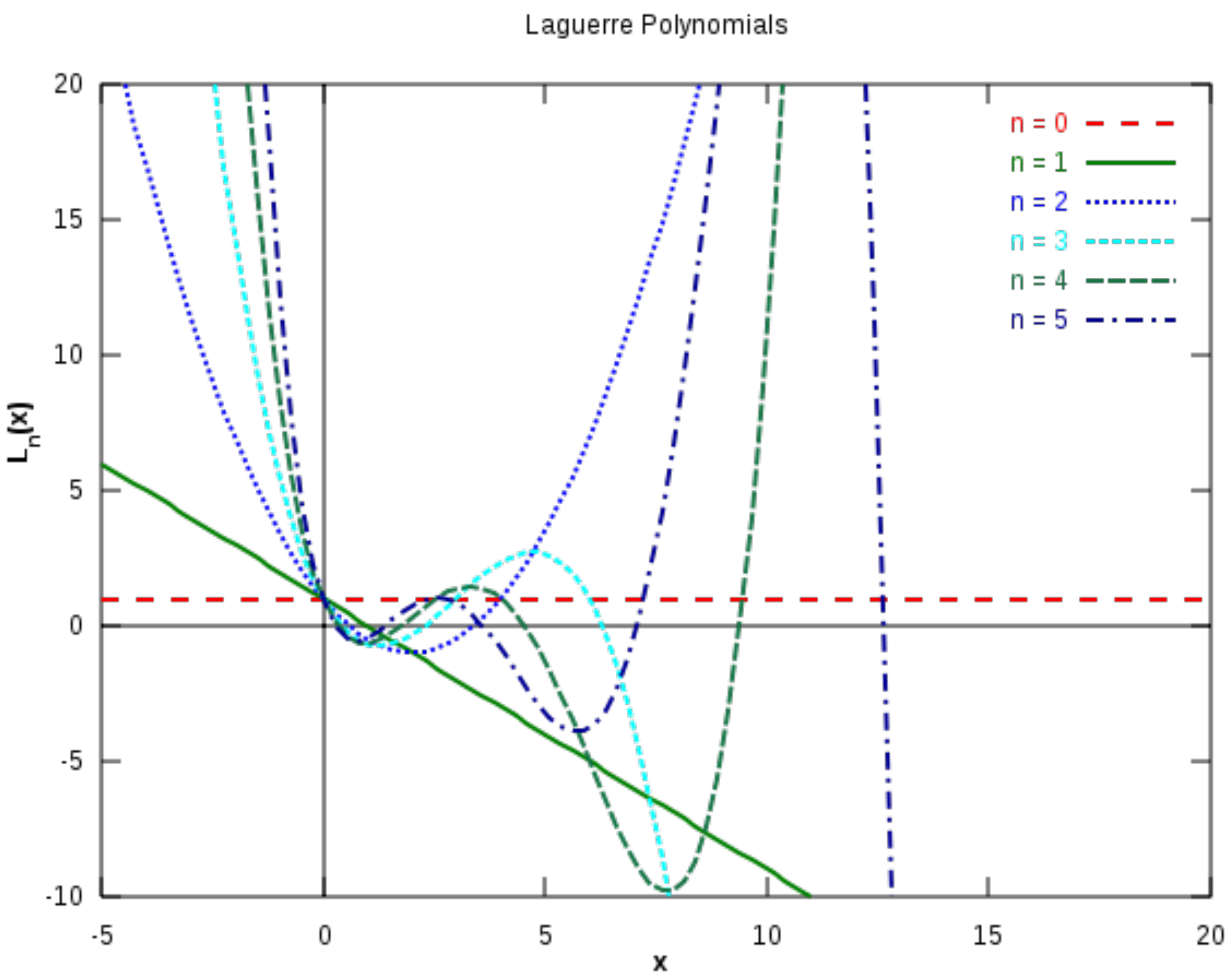
Gauss-Laguerre Quadrature

- These Laguerre polynomials form a set of polynomials with the weight function $w(x) = e^{-x}$. The quadrature rule approximates integrals of the form $\int_0^\infty f(x)e^{-x}dx$.
- This corresponds to Gauss-Laguerre Quadrature
  - $\int_0^\infty e^{-x}f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$
  - Nodes: x_i: the i-th zeros of $L_n(x)$
  - Weights: $w_i = \frac{x_i}{(n+1)^2[L_{n+1}(x_i)]^2}$
- Since the domain of integration $(0,\infty)$ is infinite, the nodes get larger and then the corresponding weights decay rapidly

Applying Gauss-Laguerre

- Example with 2 nodes/weights (method of undetermined coefficients):
- $f(x) = 1$: $\int_0^\infty 1e^{-x}dx = 1 \approx 1 \cdot w_1 + 1 \cdot w_2$
- $f(x) = x$: $\int_0^\infty xe^{-x}dx = 1 \approx x_1 \cdot w_1 + x_2 \cdot w_2$
- $f(x) = x^2$: $\int_0^\infty x^2e^{-x}dx = 2 \approx x_1^2 \cdot w_1 + x_2^2 \cdot w_2$
- $f(x) = x^3$: $\int_0^\infty x^3e^{-x}dx = 6 \approx x_1^3 \cdot w_1 + x_2^3 \cdot w_2$
- Solution: $x_1 = 2 - \sqrt{2}$, $x_2 = 2 + \sqrt{2}$, $w_1 = \frac{2+\sqrt{2}}{4}$, $w_2 = \frac{2-\sqrt{2}}{4}$

Laguerre Polynomial Graph



This figure shows a graph of the different Laguerre Polynomials with respect to different n values. You can see their corresponding roots/nodes which are,

x_i: the i-th zeros of $L_n(x)$.