The Saddle Surface

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Abstract

The saddle surface holds the overall basis for many surfaces of negative Gaussian curvature. This makes understanding the basic differential geometry properties of a saddle an important key in order to continue on to understand how other similar surfaces work. With saddles being so well known, there is much recent work and research that involves using them to explore other properties related to real-life applications. In order to understand how the differential geometry of saddles is described in this research, one must first comprehend the prior basic understanding of where saddles come from and how they function.

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1 Introduction and Prior Experience

Nearly everyone knows what a saddle is. Whether you study mathematics or not, almost everyone can picture what a saddle looks like. The Pringles potato chip or crisp is an everyday example of what a basic saddle may look like. However, most of those who teach or study mathematics know that a saddle is a generalization of a surface of negative curvature. A surface of negative curvature is a two-dimensional surface in three-dimensional Euclidean space that has negative Gaussian Curvature $k<0$ at every point. The concept of a surface of negative curvature can be generalized, for example, with respect to the dimension of the surface itself or the dimension and structure of the ambient space. Surfaces of negative curvature locally have a saddle-like structure. This means that in a sufficiently small neighborhood of any of its points, a surface of negative curvature resembles a saddle [5]. A surface is called a saddle surface if it is impossible to cut off a crust by any plane. Examples of a saddle surface are a one-sheet hyperboloid, a hyperbolic paraboloid, and a ruled surface. For a twice continuously-differentiable surface to be a saddle surface, it is necessary and sufficient that at each point of the surface its Gaussian curvature is non-positive. A surface for which all points are saddle points is a saddle surface. A saddle surface that is bounded by a rectifiable contour is, with respect to its intrinsic metric induced by the metric of the space, a two-dimensional manifold of non-positive curvature. A number of properties of surfaces of negative curvature can be generalized to the class of saddle surfaces, but it seems that these surfaces do not form such a natural class of surfaces as do convex surfaces [5].This made selecting a saddle as a capstone topic an excellent choice.

This unique quality of the saddle surface having negative Gaussian curvature is what makes a saddle so distinguishable. In some ways, one could look at a surface that has negative Gaussian curvature almost as a type of saddle since saddle surfaces hold the overall basis for having this negative curvature. This made the saddle interesting because in some aspects it holds as the original or elementary surface. For example, Enneper's surface may look like a saddle over certain parameters, however, looking at the matrix of the first fundamental form shows otherwise. Having this negative Gaussian curvature separates the saddle from convex/elliptical surfaces which have positive Gaussian curvature [5]. The saddle surface has applications in many fields of mathematics. Since there are a lot of different uses to saddles other than just in differential geometry, it's characteristics are often taught to students

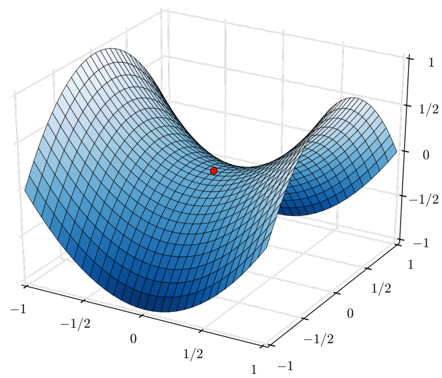

Figure 1: A saddle surface

in introductory level courses.

1.1 MAT 2240: Introduction to Linear Algebra

In MAT 2240:*Introduction to Linear Algebra* we learned the basics behind eigenvalues and how certain ones can relate to saddle points. The definitive statement for eigenvalues and eigenvectors is as follows.

Let A $\in \mathbb{C}^{nxn}$. Suppose that Ax = λx for some scalar $\lambda \in \mathbb{C}$ and nonzero vector x $\in \mathbb{C}^n$.Then λ is called an eigenvalue of A, and x is called an eigenvector of A associated with λ. Below is an example of the definition just stated.

$$\underbrace{\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{x_1} = \underbrace{1}_{\lambda_1}\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{x_1} \text{ and } \underbrace{\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{x_2} = \underbrace{3}_{\lambda_2}\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{x_2}$$

$\lambda_1 = 1$ and $\lambda_2 = 3$ are eigenvalues of A.

$$x_1 = \begin{bmatrix}1\\0\end{bmatrix}, x_2 = \begin{bmatrix}0\\1\end{bmatrix}$$

are eigenvectors associated with λ_1, λ_2.

This is meaningful because a critical point is a saddle point if the characteristic equation from a matrix has one positive and one negative eigenvalue. Having two eigenvalues of opposite signs results in a negative Gaussian curvature when they are multiplies which yields a saddle point. At such points, the surface will be saddle shaped. Which leads to the understanding that there can be more than one saddle point on a saddle surface but the main saddle point is the one that makes the basis shape of the saddle surface itself.

1.2 MAT 2130: Calculus With Analytic Geometry III

In MAT 2130: *Calculus with Analytical Geometry III* we examined and analyzed relative maximums and minimums relating to critical points that exhibit the behavior of saddle points. In order to explain what qualities of a critical point determines a saddle point, one must first understand what a critical point is. The point (a,b) is a critical point of f(x,y) provided one of the following is true,

1. ∇f(a,b) = $\vec{0}$,
2. $f_x(a,b)$ and/or $f_y(a,b)$ doesn't exist.

Now, suppose that (a,b) is a critical point of f(x,y) and that the second derivatives are continuous in some region that contains (a,b). Next, define D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2. We then have certain classifications of the critical point. The only classifications that pertain to having a saddle point at the point (a,b) are if D<0 or if D = 0. However, if D = 0 the point (a,b) may be a relative minimum, relative maximum, or a saddle point. Which means other techniques would need to be used to classify the critical point.

1.3 MAT 3130: Introduction to Differential Equations

In MAT 3130: *Introduction to Differential Equations* we explored real and distinct eigenvalues relating to a critical point being either a node or a saddle. We also looked at unstable saddle points in slope fields.

Given a general homogeneous system, $\vec{x}^t$ = A$\vec{x}$, notice that $\vec{x}$ = 0 is a solution to the system of differential equations. In the case of being restricted down to the 2 x 2 case, the system will have the form,

$$\begin{matrix} \vec{x}_1^t = ac_1 + bx_2 \\ \vec{x}_2^t = cx_1 + dx_2 \end{matrix} \Rightarrow \vec{x}^t = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$$

Solutions to this system will be of the form,

$$\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

and the equilibrium solution will be,

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

When these points are plotted, after thinking of the solutions to the system as points in the x_1-x_2 plane, the equilibrium solution will correspond to the origin of the x_1-x_2 plane and this plane is called the phase plane. Plugging chosen values of t into the solution yields a sketch to the solution in the phase plane. This is called the trajectory of the solution. Looking at whether or not the solution will approach the equilibrium solution as t increases paints a better picture of what is going on.

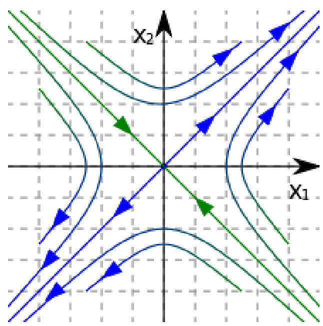

Figure 2: Phase plane of a saddle point

Figure 2 above, displays a sketch of the phase portrait of a saddle where most of the solutions start away from the equilibrium solution then as t starts to increase they move in towards the equilibrium solution and then eventually start moving away from the equilibrium solution again. In these kinds of cases, the equilibrium point is called a saddle point which is unstable since all but two of the solutions are moving away from it as t increases.

2 Historical Connections and Applications

The geometry of a saddle surface is similar to hyperbolic plane geometry, which are surfaces with a constant negative Gaussian curvature. A modern use of this hyperbolic geometry is in the theory of special relativity, particularly Minkowski spacetime and gyrovector space. However, when geometers first realized they were working with something other than the standard Euclidean geometry, they described their geometry under many different names.

Felix Klein finally gave the subject the name 'hyperbolic geometry' to include it in spherical geometry, Euclidean geometry, and hyperbolic geometry [4]. More prior progress includes Grigori Yakovlevich Perelman defending his thesis *Saddle Surfaces in Euclidean Spaces* in 1990 [1]. He had already published one of the main results of the thesis in *An example of a complete saddle surface in* $\mathbb{R}^4$ *with Gaussian curvature bounded away from zero* (Russian)(1989).

In a more recent study of connecting saddle surfaces to real life applications, Silvia Bonfanti and Walter Kob published a journal in 2017 about methods to locate saddle points in complex landscapes [2]. They offer three main approaches to find such saddle points. In the case that one knows two neighboring minima, one can use simple and efficient algorithms that can find the corresponding saddle point with a numerical approach. The second approach involves only one starting minimum and uses the geometrical information from the potential energy landscape to climb up the landscape until a saddle point is found. The last approach considers the squared gradient of the potential energy since at a saddle point $\nabla V = 0$, a minimization of the squared gradient of the potential energy will lead to a saddle point or local minimum.

3 Differential Geometry of Saddles

The basic parametrization of a saddle surface is $\vec{x}$(u,v)=(u,v,uv). A saddle surface has geodesics that consist of the cross section in the middle of the surface passing through the saddle point.

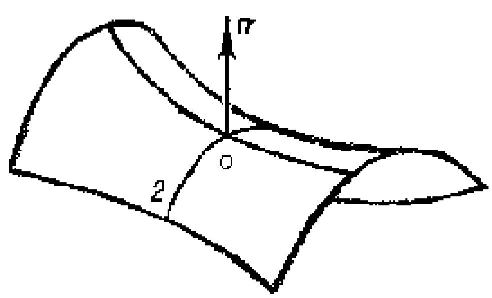

Figure 3: Geodesics on a saddle surface

It makes sense that they must nicely bisect through the saddle point because at a saddle the point the surface is saddle shaped. This leads to the fact that the saddle point is the most interesting point on a saddle. This point is typically at the origin of the surface which would be $\vec{x}$(u,v)=(0,0,0). As said before, there is negative Gaussian curvature everywhere, which includes the saddle point. Gaussian curvature determines the deviance o fa surface from being a plane at each point. This curvature for a saddle is of the form K=$\frac{-1}{(1+u^2+v^2)^2}$. This is because one of the principal curvatures is negative and the other is positive, so when they are multiplied they yield a negative Gaussian curvature. The main thing to know about having negative Gaussian curvature is that one piece of the tangent plane goes up while the other is directed down, so the surface lies on both sides of the tangent plane.

After obtaining E,F, and G from the first fundamental form, the metric form comes out to be $(\frac{ds}{dt})^2=(v^2+1)(\frac{du}{dt})^2+(2uv)(\frac{du}{dt})(\frac{dv}{dt})+(u^2+1)(\frac{dv}{dt})^2$. This metric form shows that the Pythagorean theorem does not hold because there is a nonzero F. The only way for F to be zero would if u∪v=0, and this cannot happen because then the parametrization would not be a saddle.

The rate of change of the surface normal is not a multiple of $\vec{x}_u$ or $\vec{x}_v$ individually but can be a multiple by combining the two. The reason individual multiples of $\vec{x}_u$ and $\vec{x}_v$ can't be obtained is because they both have zero components where $\vec{x}_u$=(1,0,v) and $\vec{x}_v$=(0,1,u). Therefore, there is an algebraic congruence but not an isometric one. So the covariant derivatives and shape operator would include:

$$S(\vec{x}_u)=aU_u+bU_v,$$
$$S(\vec{x}_v)=cU_u+dU_v$$

where a,b,c, and d are multiples.

4 Conclusion

The saddle surface is the overall basis of having negative curvature. This can lead to other surfaces being isometric to a saddle in different ways. For example, Enneper's surface is locally isometric but not globally. They look as if they are the same surface when viewed at certain parameters, however, after comparing their metric forms it becomes clear that they are not globally isometric.

If there's one thing to remember about the saddle surface it's the fact that it has negative Gaussian curvature. Through prior classes taken, such as Introduction to Linear Algebra, Calculus with Analytic Geometry III, and

Introduction to Differential Equations, understanding the topics in Differential Geometry pertaining to having this negative curvature were made a lot easier. After being able to understand the differential geometry behind saddles, finding current and intriguing research involving saddles became even more fascinating. If there had been more time, researching on how saddle surfaces are incorporated into general relativity would have been interesting to look into in more depth.

5 Acknowledgements and References

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